CSCE 222
Discrete Structures for Computing

Sets and Set Operations

Dr. Philip C. Ritchey

Set Notation

- **Set:** an unordered collection of objects *("members", "elements")*
- $a \in S$
  - "a is a member of S", "a is an element of S"
- $a \notin S$
  - "a is **not** a member of S", "a is **not** an element of S"
- Roster method
  - S= $\{1,2,3,4\}$
- Set Builder
  - $S = \{x \mid x \; is \; a \; positive \; integer \; less \; then \; 5\}$
  - $S = \{x \mid x \in \mathbb{Z}^{+} \land \; x < 5\}$

Exercise

- Jumping Jacks!
  - Kidding…
- List the members of the set:
  - $\{x \mid x \text{ is the square of an integer and } x < 100\}$
    - $\{0,1,4,9,16,25,36,49,64,81\}$
  - $\{x \mid x^2 = 2\}$
    - $\{-\sqrt{2}, \sqrt{2}\}$
- Use set builder notation to describe the set:
  - $\{0,3,6,9,12\}$
    - $\{3x \mid x \text{ is an integer and } 0 \leq x \leq 4\}$
  - $\{-3,-2,-1,0,1,2,3\}$
    - $\{x \mid x \text{ is an integer and } |x| \leq 3\}$

Common Sets

- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of integers
  - $\mathbb{Z}^+$, the positive integers
  - $\mathbb{Z}^-$, the negative integers
- $\mathbb{N} = \{[0,]1,2,3,\dots\}$, the set of natural numbers
- $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$, the set of rational numbers
- $\mathbb{R}$, the set of real numbers
  - $\mathbb{R}^+$, the set of positive reals
  - $\mathbb{R}^-$, the set of negative reals
- $\mathbb{C}$, the set of complex numbers
- $\boldsymbol{U}$**:** the universal set (*set of discourse*)
- $\emptyset$**:** the empty set, $\{\quad\}$
  - $\{\emptyset\}$ is a non-empty singleton set

Interval Notation

- Shortcuts for sets containing numbers
- [,] mean *inclusive*. **Closed** interval.
- (,) mean *exclusive*. **Open** interval.

$$[a, b] = \{x \mid a \leq x \leq b\}$$

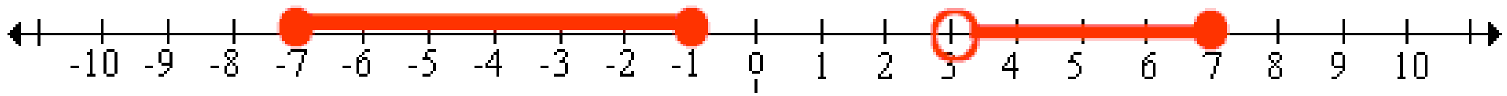
$$(a, b) = \{x \mid a < x < b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

Exercise

- Push Ups!
  - Kidding…
- Use interval notation to describe the number line:



  - $[-7, -1] \cup (3,7]$
- Express $x \neq 9$ using:
  - Set builder
    - $\{x \in \mathbb{R} \mid x \neq 9\}$
  - Interval Notation
    - $(-\infty, 9) \cup (9, \infty)$

Set Equality and Subsets

- Set Equality, $A = B$
  - $A = B \coloneqq \forall x(x \in A \leftrightarrow x \in B)$
  - $\{1,3,5\} = \{3,1,5\}$
  - $\{1,3,5\} = \{1,3,3,3,5,5,5,5,5\}$?

- Subset, $A \subseteq B$
  - $A \subseteq B \coloneqq \forall x(x \in A \rightarrow x \in B)$
  - $\{1,9\} \subseteq \{1,3,5,7,9\}$
  - $\{1,9\} \subseteq \{1,9\}$

- Proper Subset, $A \subset B$
  - $A \subset B \coloneqq A \subseteq B \land A \neq B$
  - $\{1,9\} \subset \{1,3,5,7,9\}$
  - $\{1,9\} \not\subset \{1,9\}$

- $\emptyset \subseteq A$, for any set A
- $A \subseteq A$, for any set A

- How can we express equality in terms of subsets?
  - $A = B \leftrightarrow (A \subseteq B \land B \subseteq A)$

Set Size and Power Sets

- Set size
  - The number of distinct elements in the set
  - $|S| = n,\ n \geq 0, n$ is **finite**
  - **“cardinality”**
- Power set
  - The set of all subsets of a set
  - $\wp(S)$
  - Ex: $S = \{1,2,3\}$
    - $\wp(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
  - $|\wp(S)| = 2^{|S|}$

Cartesian Products

- $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
- $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i\}$
  - $(a_1, a_2, \dots, a_n)$ is an n-tuple
- $A^2 = A \times A$
  - $A^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A\}$
- $A \times B \neq B \times A$
  - Exceptions: $A = \emptyset,\ B = \emptyset,\ A = B$
- A subset of a Cartesian product is called a *Relation* from A to B
  - We will cover Relations soon.

Exercise

- Let A be the set of all airlines and B be the set of all US cities.
  - What is the Cartesian product $A \times B^2$?
    - $\{(a, b, c) \mid a\ is\ an\ airline, b\ is\ a\ US\ city\ and\ c\ is\ a\ US\ city\}$
  - How can this Cartesian product be used?
    - Each element is a route that an airline flies, from one city to another
    - Could be used for route planning
- Does $(A \times B) \times (C \times D) = A \times (B \times C) \times D$?
  - No. $\{((a, b), (c, d))\} \neq \{(a, (b, c), d)\}$

Set Operations

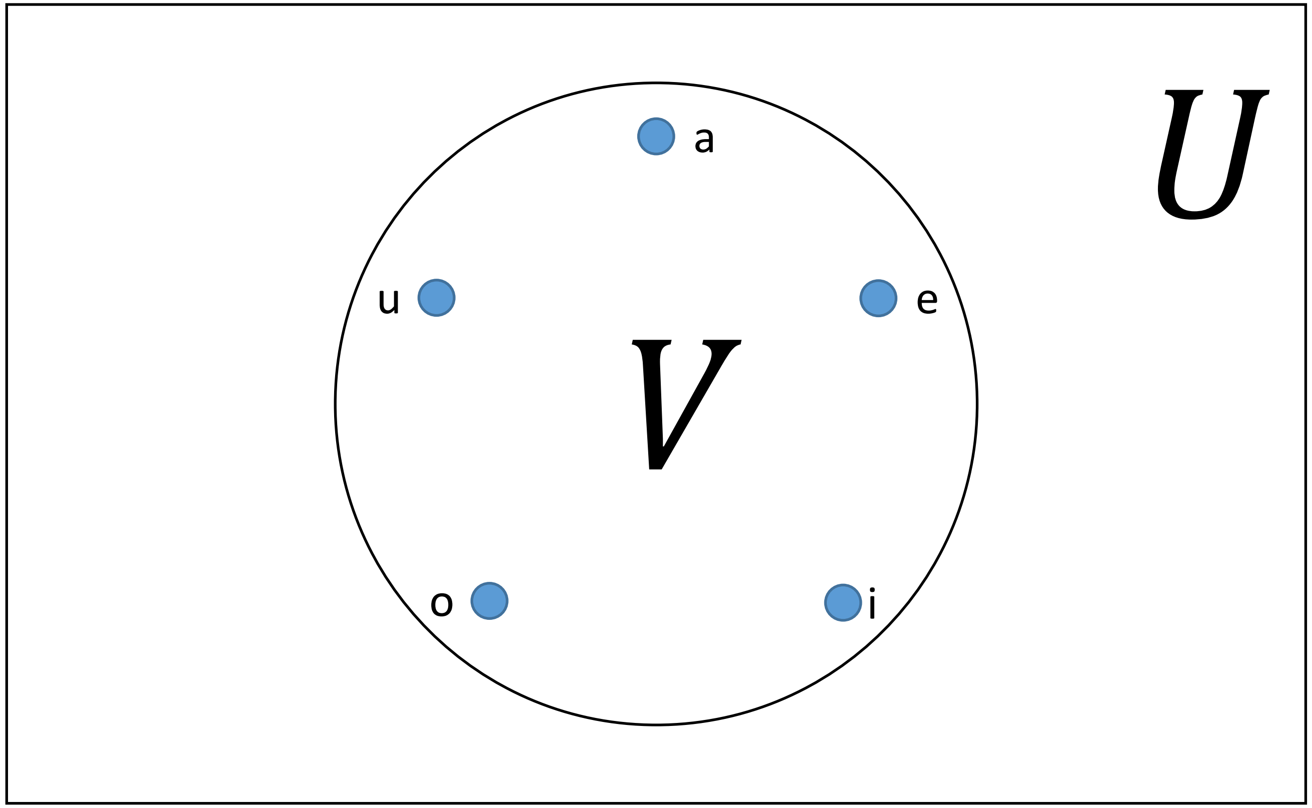
- Set Union, $A \cup B$
  - $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - $\{1,2,3\} \cup \{3,4\} = \{1,2,3,4\}$
- Set Intersection, $A \cap B$
  - $A \cap B = \{x \mid x \in A \wedge x \in B\}$
  - $\{1,2,3\} \cap \{3,4\} = \{3\}$
  - If $A \cap B = \emptyset$, then A and B are *disjoint*.
- Principle of inclusion-exclusion
  - $|A \cup B| = |A| + |B| - |A \cap B|$

Set Operations

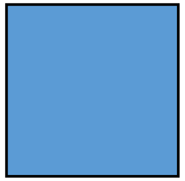
- Set Difference, $A - B$
  - $A - B = \{x \mid x \in A \land x \notin B\}$
  - $\{8,6,7,5,3,0,9\} - \{0,2,4,6,8\} = ?$
- Complement, $\bar{A}$
  - $\bar{A} = \{x \mid x \notin A\}$
  - $\bar{A} = U - A$
- How can we express set difference using the other set operations?
  - $A - B = A \cap \bar{B}$

Venn Diagrams

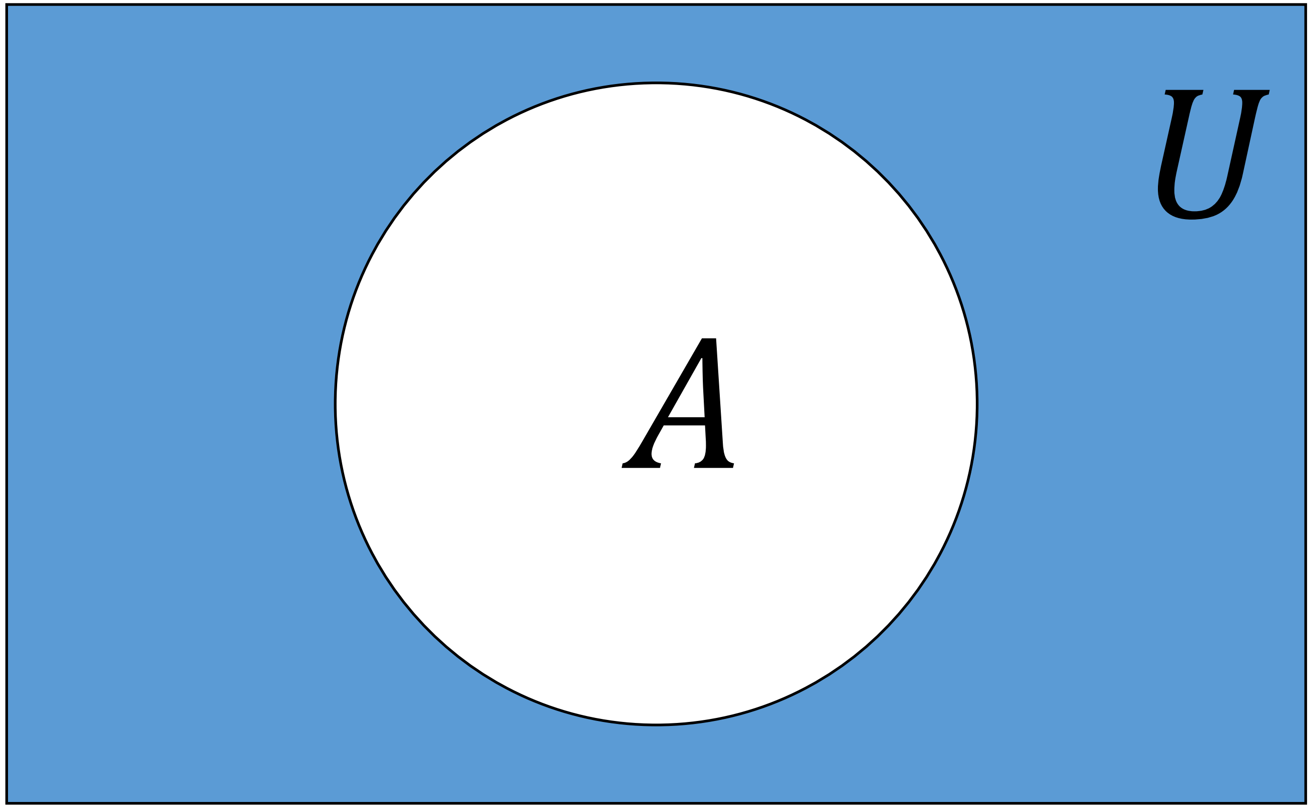
- Draw a rectangle to represent U.
- Draw geometric shapes inside to represent sets.
- Use points to represent particular elements.



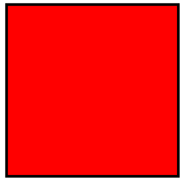
Venn Diagrams



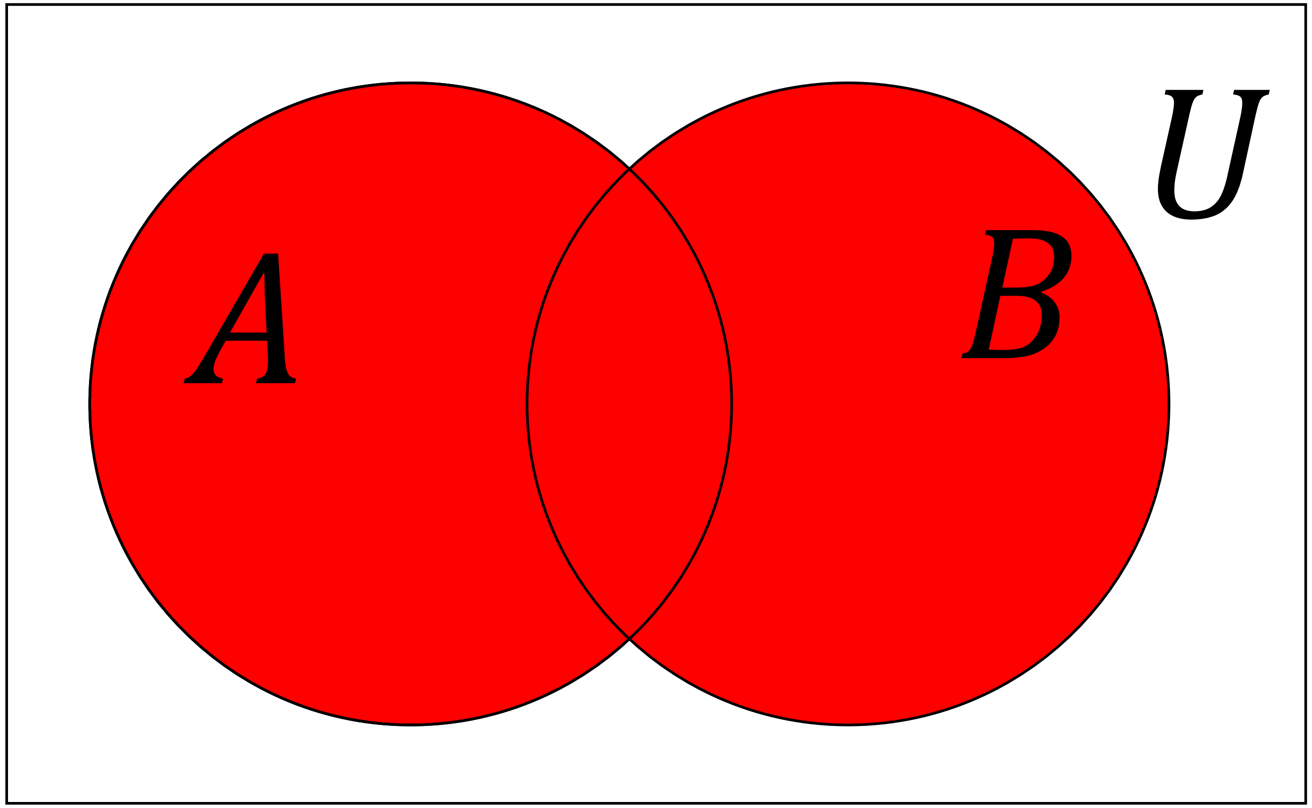
$= \bar{A}$



Venn Diagrams

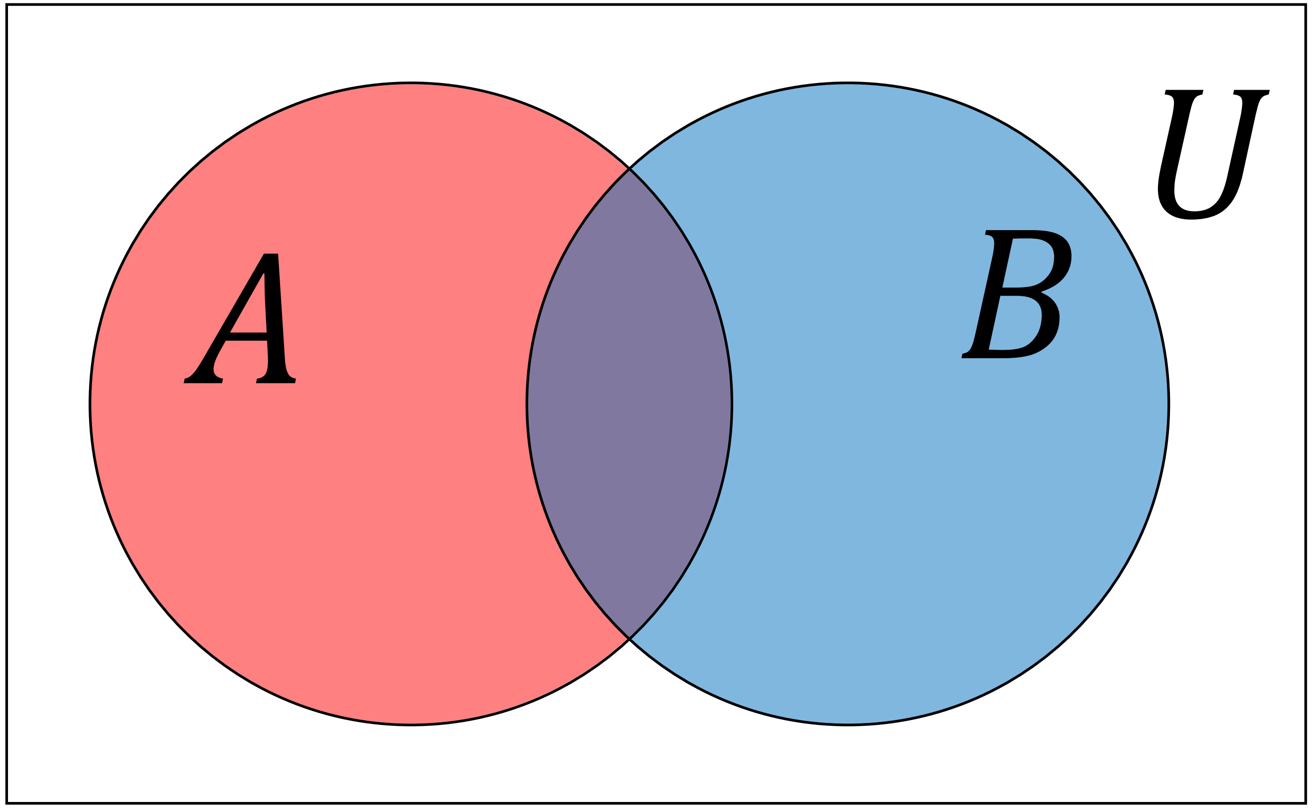


$= A \cup B$

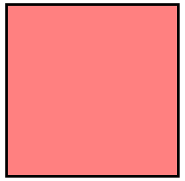


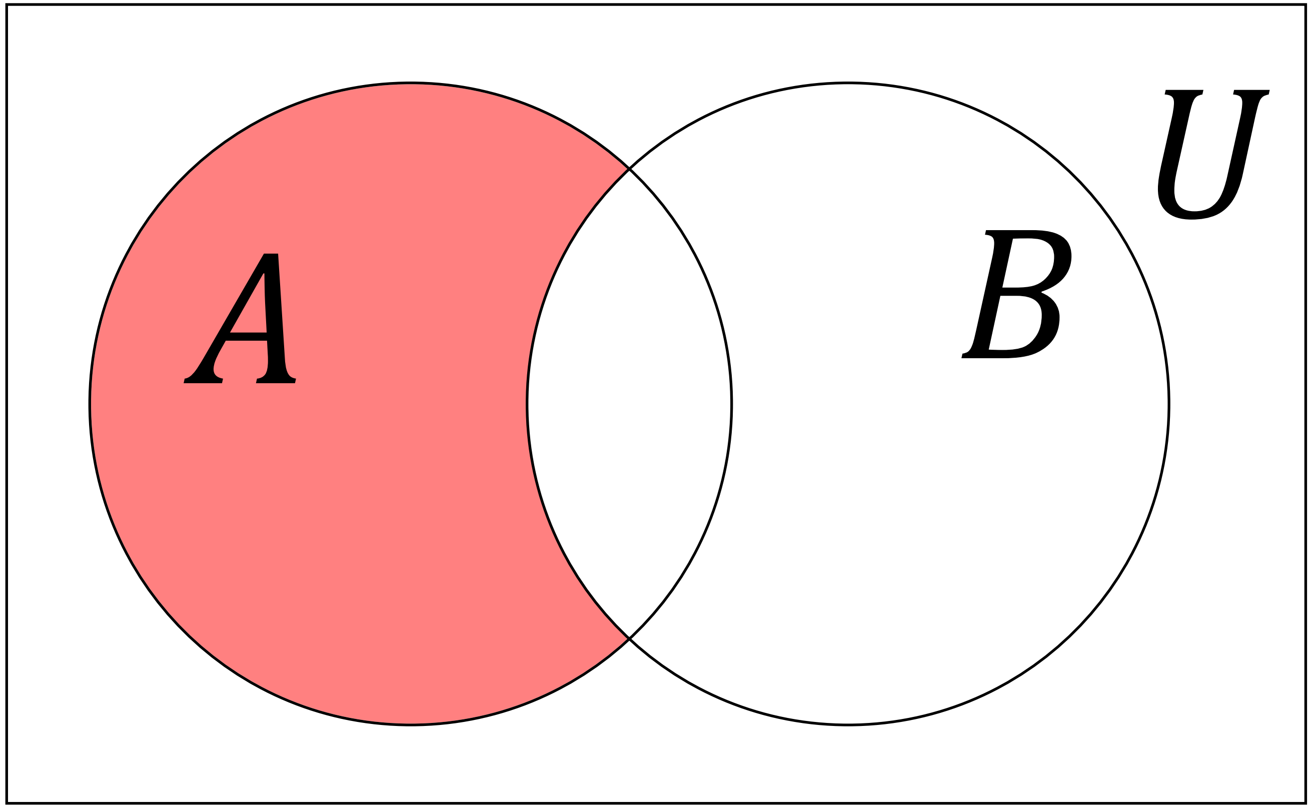
Venn Diagrams

$\square = A \cap B$



Venn Diagrams

 $= A - B$



Set Identities

- Same as for propositional logic!
  - Identity, domination, idempotent, double negation, commutativity, associativity, distributivity, DeMorgan, absorption, negation
- Operators map from logic to sets
  - $p \vee q \Rightarrow P \cup Q$
  - $p \wedge q \Rightarrow P \cap Q$
  - $\neg p \Rightarrow \bar{P}$
  - $p \rightarrow q \Rightarrow P \subseteq Q$
  - $p \leftrightarrow q \Rightarrow P = Q$

Set Identities

- Ex: $\overline{A \cap B} = \bar{A} \cup \bar{B}$ (DeMorgan)
  - Show $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$
    - Show $x \in \overline{A \cap B} \rightarrow x \in (\bar{A} \cup \bar{B})$
    - Assume $x \in \overline{A \cap B}$
    - Then, $x \notin (A \cap B)$
    - $\neg(x \in A \cap B)$
    - $\neg(x \in A \wedge x \in B)$
    - $\neg(x \in A) \vee \neg(x \in B)$
    - $(x \notin A) \vee (x \notin B)$
    - $x \in \bar{A} \vee x \in \bar{B}$
    - $x \in \bar{A} \cup \bar{B}$
    - $\square$
  - Show $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$
    - Show $x \in (\bar{A} \cup \bar{B}) \rightarrow x \in \overline{A \cap B}$
    - Assume $x \in (\bar{A} \cup \bar{B})$
    - Then, $x \in \bar{A} \vee x \in \bar{B}$
    - $(x \notin A) \vee (x \notin B)$
    - $\neg(x \in A) \vee \neg(x \in B)$
    - $\neg(x \in A \wedge x \in B)$
    - $\neg(x \in A \cap B)$
    - $x \notin (A \cap B)$
    - $x \in \overline{A \cap B}$
    - $\square$

Truth tables ⇒ Membership tables

Verify that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{A \cap B}$	$\boldsymbol{\overline{A \cap B}}$	$\boldsymbol{\bar{A}}$	$\boldsymbol{\bar{B}}$	$\boldsymbol{\bar{A} \cup \bar{B}}$
0	0	0	**1**	1	1	**1**
0	1	0	**1**	1	0	**1**
1	0	0	**1**	0	1	**1**
1	1	1	**0**	0	0	**0**

XOR

- $A \oplus B$: elements in A, or in B, but not in both.
  - Symmetric difference

- Draw the Venn diagram

- Claim: $A \oplus B = (A \cup B) - (A \cap B)$
  - Proof?